

# A COMPARISON PRINCIPLE FOR SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH JUMPS

KONSTANTINOS-ANASTASIOS DAREIOTIS AND ISTVAN GYÖNGY

**ABSTRACT.** We prove the comparison principle for semilinear stochastic partial differential equations with jumps on the whole space and on bounded Lipschitz domains. The results are a consequence of an Itô's formula for the positive part.

## 1. INTRODUCTION

Our goal is to prove a comparison principle for semilinear stochastic partial differential equations with jumps. For this, we obtain in section 2 an Itô's formula for the positive part. Then we use it in section 3 to prove our main result. The comparison principle is a very powerful tool and there are many results on it in PDE theory. In SPDE theory, comparison theorems can be found in [9], [4], [10], [2], [1] and [3]. In [10] a maximum principle is proved for linear equations without jumps, on  $C^1$  domains. Also a comparison principle is proved in [2] for quasilinear equations without jumps, but in a different framework.

Let us now introduce our setting. We use a similar approach to [10]. We consider a complete probability space  $(\Omega, \mathcal{F}, P)$  equipped with a right-continuous filtration  $(\mathcal{F}_t)_{t \geq 0}$ , such that  $\mathcal{F}_0$  contains all  $P$ -zero sets. We consider also a  $\sigma$ -finite measure space  $(Z, \mathcal{Z}, \nu)$  and a quasi left-continuous point process  $(p_t)_{t \in [0, T]}$  in  $Z$  for a finite  $T > 0$ . Let  $N(dt, dz)$  be the corresponding random measure on  $[0, T] \times Z$ . We assume that its compensator is  $\nu(dz)dt$ , and use the notation

$$\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt.$$

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Let  $\{w_t^k\}_{k=1}^\infty$  be a sequence of independent real valued  $\mathcal{F}_t$ -Wiener processes. We will consider the equation

$$\begin{aligned} du_t(x) = & \{\mathcal{L}_t u_t(x) + f_t(x, u_t(x))\} dt \\ & + \{\phi_t^{ik}(x) D_i u_t(x) + \sigma_t^k(x, u_t(x))\} dw_t^k \\ & + \int_Z g_t(x, z, u_t(x)) \tilde{N}(dt, dz), \end{aligned} \quad (1.1)$$

for  $(t, x) \in [0, T] \times Q$ , with initial condition

$$u_0(x) = \psi(x), \quad x \in Q, \quad (1.2)$$

where  $Q$  is either the whole  $\mathbb{R}^d$  or a bounded Lipschitz domain in  $\mathbb{R}^d$  and

$$\mathcal{L}_t u = D_j(a_t^{ij}(x) D_i u) + b_t^i(x) D_i u + c_t(x) u, \quad D_i = \frac{\partial}{\partial x^i}, \quad i = 1, \dots, d.$$

Here and later on, unless otherwise indicated, the summation convention is used with respect to repeated integer valued indices. For notions and basic results on point processes and on the related stochastic integrals we refer to [8] and [12].

In conclusion we introduce some notation used throughout the paper. If  $X$  is a topological space then  $\mathcal{B}(X)$  is the Borel  $\sigma$ -algebra on  $X$ . The notation  $\mathcal{P}$  is used for the predictable  $\sigma$ -algebra on  $\Omega \times [0, T]$ . If  $X$  is a normed linear space then  $|x|_X$  denotes the norm of  $x \in X$ . If  $X^*$  is the dual of  $X$  then  $\langle x, x^* \rangle$  denotes the action of  $x^* \in X^*$  on  $x \in X$ . If  $X$  is a Hilbert space we write  $(\cdot, \cdot)$  for the inner product in  $X$ . As usual we denote by  $H^1(Q)$  the space of functions  $u \in L^2(Q)$ , such that the distributional derivatives of first order lie in  $L^2(Q)$ . We write  $H_0^1(Q)$  for the closure of  $C_c^\infty(Q)$  in  $H^1(Q)$  under the norm

$$|u|_{H^1} = \left( \sum_{i=1}^d |D_i u|_{L^2}^2 + |u|_{L^2}^2 \right)^{1/2}.$$

We will use the notation  $H^{-1}(Q)$  for the dual of  $H_0^1(Q)$ . Finally, for a real number  $r$ , we write  $r^+$  for its positive part.

## 2. ITÔ'S FORMULA FOR THE SQUARE OF THE NORM OF THE POSITIVE PART

We are interested in an Itô's formula for  $|u_t^+|_{L^2(Q)}^2$ , where  $u_t$  is an  $H^{-1}(Q)$ -valued semimartingale taking values in  $H_0^1(Q)$  for  $P \times dt$  almost every  $(\omega, t) \in \Omega \times [0, T]$ . To state the formula we set

$$V := H_0^1(Q), \quad H := L^2(Q), \quad V^* := H^{-1}(Q),$$

and we consider the processes

$$v : \Omega \times [0, T] \rightarrow V, \quad v^* : \Omega \times [0, T] \rightarrow V^*, \quad h^k : \Omega \times [0, T] \rightarrow H,$$

$$K : \Omega \times [0, T] \times Z \rightarrow H,$$

for integers  $k \geq 1$ , where  $v$  is progressively measurable,  $v^*$  and  $h^k$  are  $\mathcal{F}_t$ -adapted, and  $K$  is  $\mathcal{P} \times \mathcal{Z}$  measurable. We consider also  $\psi$ , an  $\mathcal{F}_0$ -measurable random variable in  $H$  with  $E|\psi|_H^2 < \infty$ . We make the following assumption.

**Assumption 2.1.**

i) Almost surely

$$\int_0^T \left( |v_t|_V^2 + |v_t^*|_{V^*}^2 + \sum_k |h_t^k|_H^2 + \int_Z |K_t(z)|_H^2 \nu(dz) \right) dt < \infty,$$

ii)  $\nu(Z) < \infty$ ,

iii) for each  $\phi \in V$  we have for  $dP \times dt$ -almost every  $(\omega, t)$ ,

$$(u_t, \phi) = (\psi, \phi) + \int_0^t \langle v_s^*, \phi \rangle ds + \int_0^t (h_s^k, \phi) dw_s^k$$

$$+ \int_0^t \int_Z (K_s(z), \phi) \tilde{N}(ds, dz).$$

**Theorem 2.1.** *Suppose that Assumption 2.1 is satisfied. Then there exist a set  $\tilde{\Omega} \subset \Omega$  of probability one, and an  $H$ -valued strongly cadlag adapted process  $u_t$  such that  $u_t = v_t$  for  $dP \times dt$ -almost every  $(\omega, t)$ . Moreover for  $\omega \in \tilde{\Omega}$ ,  $t \in [0, T]$  we have*

$$i) u_t = \psi + \int_0^t v_s^* ds + \int_0^t h_s^k dw_s^k + \int_0^t \int_Z K_s(z) \tilde{N}(ds, dz), \quad (2.1)$$

$$ii) |u_t^+|_H^2 = |\psi^+|_H^2 + 2 \int_0^t \langle v_s^*, u_s^+ \rangle ds + 2 \int_0^t (h_s^k, u_s^+) dw_s^k$$

$$+ \int_0^t \sum_k |\mathbb{I}_{\{u_s > 0\}} h_s^k|_H^2 ds + \int_0^t \int_Y |(u_{s-} + K_s(z))^+|_H^2 - |u_{s-}^+|_H^2 \tilde{N}(ds, dz)$$

$$+ \int_0^t \int_Y |(u_s + K_s(z))^+|_H^2 - |u_s^+|_H^2 - 2(K_s(z), u_s^+) \nu(dz) ds.$$

To prove Theorem 2.1 we need two lemmas.

**Lemma 2.2.** *Let  $(X, \Sigma, \mu)$  be a measure space, and let  $u_n, u \in L^1(X)$ . Suppose that  $u_n \rightarrow u$  in  $L^1(X)$ . Then there exists a subsequence  $u_{n(k)}$  and a function  $v \in L^1(X)$  such that for all  $x \in X$ , we have  $|u_{n(k)}(x)| \leq v(x)$ .*

*Proof.* There exists a subsequence  $u_{n(k)}$  such that

$$|u_{n(k)} - u|_{L^1} \leq 1/2^k.$$

Set  $v(x) = |u(x)| + \sum_k |u_{n(k)}(x) - u(x)|$ . Then  $v$  has the desired properties.  $\square$

The next lemma is from [2].

**Lemma 2.3.** *Let  $\mathcal{Q}$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ . Take  $\phi_n \in C_c^\infty(\mathcal{Q})$  with*

$$(i) \ 0 \leq \phi_n \leq 1$$

$$(ii) \ \phi_n = 1 \text{ on } \{x \in \mathcal{Q}, r(x) \geq 1/n\}$$

$$(iii) \ \phi_n = 0 \text{ on } \{x \in \mathcal{Q}, r(x) \leq 1/2n\}$$

$$(iv) \ |(\phi_n)_{x_i}| \leq Cn, \text{ where } C \text{ is a constant and } r(x) = \text{dist}(x, \partial\mathcal{Q}).$$

*Then  $\phi_n v \rightarrow v$  in  $H_0^1(\mathcal{Q})$  for all  $v \in H_0^1(\mathcal{Q})$ , and for some constant  $C$  we have*

$$\sup_n |\phi_n v|_{H_0^1} \leq C|v|_{H_0^1}, \quad \forall v \in H_0^1(\mathcal{Q}).$$

*Remark 2.1.* One can easily see the existence of such sequence. We note that for such  $\phi_n$ ,  $\phi_n^2$  satisfies again (i)-(iv), and this implies that  $\phi_n^2 v \rightarrow v$  in  $H_0^1(\mathcal{Q})$ , for all  $v \in H_0^1(\mathcal{Q})$ , and for some constant  $k$  we have

$$\sup_n |\phi_n^2 v|_{H_0^1} \leq k|v|_{H_0^1}, \quad \forall v \in H_0^1(\mathcal{Q}).$$

We introduce now the functions  $\alpha_\delta(r)$ ,  $\beta_\delta(r)$  and  $\gamma_\delta(r)$  on  $\mathbb{R}$  given by

$$a_\delta(r) = \begin{cases} 1 & \text{if } r > \delta \\ \frac{r}{\delta} & \text{if } 0 \leq r \leq \delta \\ 0 & \text{if } r < 0 \end{cases}$$

$$\beta_\delta(r) = \int_0^r a_\delta(s) ds, \quad \gamma_\delta(r) = \int_0^r \beta_\delta(s) ds.$$

For all  $r \in \mathbb{R}$  we have  $\alpha_\delta(r) \rightarrow \mathbb{I}_{r>0}$ ,  $\beta_\delta(r) \rightarrow r^+$  and  $\gamma_\delta(r) \rightarrow (r^+)^2/2$  as  $\delta \rightarrow 0$ . Also the following inequalities hold

$$|\alpha_\delta(r)| \leq 1, \quad |\beta_\delta(r)| \leq |r|, \quad |\gamma_\delta(r)| \leq \frac{r^2}{2}.$$

We are now ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* We only prove ii) since the rest of the assertions are proved in [5]. Notice that by using a standard localization argument we can assume that

$$E \int_0^T \left( |u_s|_V^2 + |v_t^*|_{V^*}^2 + \sum_k |h_t^k|_H^2 + \int_Z |K_t(z)|_H^2 \nu(dz) \right) dt < \infty$$

First we prove the statement when  $Q = \mathbb{R}^d$ . We have that equality 2.1 is satisfied iff for all  $v \in V$ , almost surely, for all  $t$  we have

$$\begin{aligned} (u_t, v) &= (u_0, v) + \int_0^t \langle v_s^*, v \rangle ds + \int_0^t (h_s^k, v) dw_s^k \\ &\quad + \int_0^t \int_Z (K_s(z), v) \tilde{N}(ds, dz). \end{aligned} \quad (2.2)$$

Let  $\phi(y)$  be a molifier with compact support. For fixed  $x$ , the function  $\phi_\epsilon(x - \cdot)$  is in  $V$ , so we can plug it in (2.2) instead of  $v$ , to get that almost surely for all  $t$

$$\begin{aligned} u_t^\epsilon(x) &= u_0^\epsilon(x) + \int_0^t v_s^{*\epsilon}(x) ds + \int_0^t h_s^{k\epsilon}(x) dw_s^k \\ &\quad + \int_0^t \int_Z K_s^\epsilon(z, x) \tilde{N}(ds, dz). \end{aligned}$$

Note that  $u_{0\epsilon}$  is  $\mathcal{F}_0 \times \mathcal{B}(\mathbb{R}^d)$  measurable. Also  $u^\epsilon, v^{*\epsilon}$  and  $h^{k\epsilon}$  are jointly measurable in  $(t, \omega, x)$ ,  $\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)$  measurable for each  $t$ , and  $K^\epsilon$  is  $\mathcal{P} \times \mathcal{Z} \times \mathcal{B}(\mathbb{R}^d)$  measurable. It is also easy to see that there exists a constant  $C_\epsilon$ , depending on  $\epsilon$ , such that for all  $t, \omega, x, z$

$$\begin{aligned} |u_t^\epsilon(x)| &\leq C_\epsilon |u_t|_H, \quad |u_0^\epsilon(x)| \leq C_\epsilon |u_0|_H, \quad |v_t^{*\epsilon}|_H \leq C_\epsilon |v_t^*|_{V^*} \\ |v_t^{*\epsilon}(x)| &\leq C_\epsilon |v_t^*|_{V^*}, \quad |h_t^{k\epsilon}(x)| \leq C_\epsilon |h_s^k|_H, \\ |K_t^\epsilon(x, z)| &\leq |K_t(z)|_H \end{aligned} \quad (2.3)$$

One can also check that for a constant  $C$ , for all  $\epsilon$

$$\begin{aligned} |u_t^\epsilon|_H &\leq C |u_t|_H, \quad |u_0^\epsilon|_H \leq C |u_0|_H, \\ |h_t^{k\epsilon}|_H &\leq C |h_s^k|_H, \quad |v_t^{*\epsilon}|_{V^*} \leq C |v_t^*|_{V^*}, \quad |u_t^\epsilon|_V \leq C |u_t|_V. \end{aligned} \quad (2.4)$$

Now let  $\alpha_\delta, \beta_\delta, \gamma_\delta$  be as before, and fix  $x$ . By Itô's formula (see [8] or [12]), we have that for each  $x$  there exists a set  $\Omega_x$  of full probability, such that for all  $\omega \in \Omega_x$  and  $t \in [0, T]$  we have

$$\begin{aligned} \gamma_\delta(u_t^\epsilon(x)) &= \gamma_\delta(u_0^\epsilon(x)) + \int_0^t \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x) ds \\ &\quad + \sum_k \int_0^t \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x) dw_s^k + \frac{1}{2} \sum_k \int_0^t \alpha_\delta(u_s^\epsilon(x)) |h_s^{k\epsilon}(x)|^2 ds \\ &\quad + \int_{(0,t]} \int_Z \gamma_\delta(u_{s-}^\epsilon(x) + K_s^\epsilon(z, x)) - \gamma_\delta(u_{s-}^\epsilon(x)) \tilde{N}(ds, dz) \\ &\quad + \int_0^t \int_Z \gamma_\delta(u_s^\epsilon(x) + K_s^\epsilon(z, x)) \\ &\quad - \gamma_\delta(u_s^\epsilon(x)) - \beta_\delta(u_s^\epsilon(x)) K_s^\epsilon(z, x) \nu(dz) ds. \end{aligned} \quad (2.5)$$

If necessary, one can redefine the sum of stochastic integrals, such that the above equality holds for all  $(t, \omega, x)$ . We would like now to integrate the above equality over  $\mathbb{R}^d$  and change the order of the integrals. For the deterministic integrals this is obvious. For the sum of integrals against the Wiener processes, since  $|\beta_\delta(x)| \leq |x|$ , according to Lemma 2.6 in [11] it suffices to show

$$E \int_{\mathbb{R}^d} \left( \int_0^T \sum_k |u_t^\epsilon(x) h_t^{k\epsilon}(x)|^2 dt \right)^{1/2} dx < \infty.$$

To check this, we calculate

$$\begin{aligned} & E \int_{\mathbb{R}^d} \left( \int_0^T \sum_k |u_t^\epsilon(x) h_t^{k\epsilon}(x)|^2 dt \right)^{1/2} dx \\ & \leq E \int_{\mathbb{R}^d} \sup_{t \leq T} |u_t^\epsilon(x)| \left( \int_0^T \sum_k |h_t^{k\epsilon}(x)|^2 dt \right)^{1/2} dx \\ & \leq E \int_{\mathbb{R}^d} \sup_{t \leq T} |u_t^\epsilon(x)|^2 dx + E \int_0^T \sum_k |h_t^{k\epsilon}|_H^2 dt \end{aligned}$$

The last integral is finite, so we only need to show that

$$E \int_{\mathbb{R}^d} \sup_{t \leq T} |u_t^\epsilon(x)|^2 dx < \infty.$$

To show this, we first prove that  $E \sup_{t \leq T} |u_t^\epsilon(x)|^2 < \infty$ . Using Burkholder's inequality we get

$$\begin{aligned} E \sup_{t \leq T} |u_t^\epsilon(x)|^2 & \leq N \left\{ E |u_0^\epsilon|^2 + E \int_0^T |v_t^{*\epsilon}(x)|^2 dt + E \left( \int_0^T \sum_k |h_t^{k\epsilon}(x)|^2 dt \right)^{1/2} \right. \\ & \quad \left. + E \left( \int_0^T \int_Z |K_t^\epsilon(z, x)|^2 \nu(dz) dt \right)^{1/2} \right\} \\ & \leq NC_\epsilon \left\{ E |u_0|^2_H + E \int_0^T |v_t^*|_{V^*}^2 dt + E \int_0^T \sum_k |h_t^k|_H^2 dt \right. \\ & \quad \left. + E \int_0^T \int_Z |K_t^\epsilon(z, x)|^2 \nu(dz) dt + 2 \right\} < \infty \end{aligned}$$

with a constant  $N$ . By Itô's formula again, taking suprema and expectations, and then using Burkholder's inequality only for the sum of the integrals against the Wiener processes we get

$$E \sup_{t \leq T} |u_t^\epsilon(x)|^2 \leq E |u_0^\epsilon(x)|^2 + 2E \int_0^T |u_s^\epsilon(x) v_s^{*\epsilon}(x)| ds$$

$$\begin{aligned}
& NE \left( \int_0^T \sum_k |u_s^\epsilon(x) h_s^{k\epsilon}(x)|^2 ds \right)^{1/2} + E \int_0^T \sum_k |h_s^{k\epsilon}(x)|^2 ds \\
& + E \sup_{t \leq T} \left| \int_{(0,t]} \int_Z 2u_{s-}^\epsilon(x) K_s^\epsilon(z, x) + |K_s^\epsilon(z, x)|^2 \tilde{N}(ds, dz) \right| \\
& \quad + E \int_0^T \int_Z |K_s^\epsilon(z, x)|^2 \nu(dz) ds \\
& \leq E |u_0^\epsilon(x)|^2 + E \int_0^T |u_s^\epsilon(x)|^2 ds + E \int_0^T |v_s^{*\epsilon}(x)|^2 ds \\
& NE \sup_{t \leq T} |u_s^\epsilon(x)| \left( \int_0^T \sum_k |h_s^{k\epsilon}(x)|^2 ds \right)^{1/2} + E \int_0^T \sum_k |h_s^{k\epsilon}(x)|^2 ds \\
& + E \sup_{t \leq T} \left| \int_{(0,t]} \int_Z 2u_{s-}^\epsilon(x) K_s^\epsilon(z, x) + |K_s^\epsilon(z, x)|^2 \tilde{N}(ds, dz) \right| \\
& \quad + E \int_0^T \int_Z |K_s^\epsilon(z, x)|^2 \nu(dz) ds.
\end{aligned}$$

Hence for any  $a > 0$

$$\begin{aligned}
E \sup_{t \leq T} |u_t^\epsilon(x)|^2 & \leq E |u_0^\epsilon(x)|^2 + E \int_0^T |u_s^\epsilon(x)|^2 ds + E \int_0^T |v_s^{*\epsilon}(x)|^2 ds \\
& + NaE \sup_{t \leq T} |u_t^\epsilon(x)|^2 + \frac{N' + a}{a} E \int_0^T \sum_k |h_s^{k\epsilon}(x)|^2 ds \\
& + E \sup_{t \leq T} \left| \int_{(0,t]} \int_Z 2u_{s-}^\epsilon(x) K_s^\epsilon(z, x) + |K_s^\epsilon(z, x)|^2 \tilde{N}(ds, dz) \right| \\
& \quad + E \int_0^T \int_Z |K_s^\epsilon(z, x)|^2 \nu(dz) ds.
\end{aligned}$$

For the integral against the compensated measure we have

$$\begin{aligned}
& E \sup_{t \leq T} \left| \int_{(0,t]} \int_Y 2u_{s-}^\epsilon(x) K_s^\epsilon(z, x) + |K_s^\epsilon(z, x)|^2 \tilde{N}(ds, dz) \right| \\
& \leq E \int_{(0,T]} \int_Z 2|u_{s-}^\epsilon(x) K_s^\epsilon(z, x)| + |K_s^\epsilon(z, x)|^2 N(ds, dz) \\
& \quad + E \int_0^T \int_Z 2|u_s^\epsilon(x) K_s^\epsilon(z, x)| + |K_s^\epsilon(z, x)|^2 \nu(dz) ds. \\
& = 2E \int_0^T \int_Z 2|u_s^\epsilon(x) K_s^\epsilon(z, x)| + |K_s^\epsilon(z, x)|^2 \nu(dz) ds \\
& \leq 2\nu(Z)E \int_0^T |u_s^\epsilon(x)|^2 ds + 4 \int_0^T \int_Y |K_s^\epsilon(z, x)|^2 \nu(dz) ds.
\end{aligned}$$

Summarizing the above we have

$$\begin{aligned} E \sup_{t \leq T} |u_t^\epsilon(x)|^2 &\leq E |u_0^\epsilon(x)|^2 + (1 + 2\nu(Z)) E \int_0^T |u_s^\epsilon(x)|^2 ds \\ &\quad + E \int_0^T |v_s^{*\epsilon}(x)|^2 ds + N' a E \sup_{t \leq T} |u_s^\epsilon(x)|^2 \\ &\quad + \frac{N' + a}{a} E \int_0^T \sum_k |h_s^{k\epsilon}(x)|^2 ds + 5 \int_0^T \int_Z |K_s^\epsilon(z, x)|^2 \nu(dz) ds. \end{aligned}$$

Choosing  $a$  small enough we have that for some constant  $\varkappa$  we have

$$\begin{aligned} E \sup_{t \leq T} |u_t^\epsilon(x)|^2 &\leq \varkappa \left\{ E |u_0^\epsilon(x)|^2 + E \int_0^T |u_t^\epsilon(x)|^2 dt \right. \\ &\quad \left. + E \int_0^T |v_t^{*\epsilon}(x)|^2 dt + E \int_0^T \sum_k |h_t^{k\epsilon}(x)|^2 dt + E \int_0^T \int_Z |K_s^\epsilon(z, x)|^2 \nu(dz) ds \right\}. \end{aligned}$$

Integrating the above relation over  $\mathbb{R}^d$ , and using Fubini's theorem and the relations in (2.3), (2.4) we conclude that for some constant  $C_\epsilon$  we have

$$\begin{aligned} E \int_{\mathbb{R}^d} \sup_{t \leq T} |u_t^\epsilon(x)|^2 dx &\leq C_\epsilon \left\{ E |u_0|_H^2 + E \int_0^T |u_t|_H^2 + |v_t^*|_{V^*}^2 dt \right. \\ &\quad \left. + E \int_0^T \sum_k |h_s^k|_H^2 dt + E \int_0^T \int_Z |K_\epsilon(s, z)|_H^2 \nu(dz) ds \right\} < \infty, \end{aligned}$$

showing that we can interchange the sum and the integrals at the sum of the stochastic integrals against the Wiener processes. For the integral against  $\tilde{N}$  in (2.5) we have

$$\begin{aligned} &\int_{(0,t]} \int_Z \gamma_\delta(u_{s-}^\epsilon(x) + K_s^\epsilon(z, x)) - \gamma_\delta(u_{s-}^\epsilon(x)) \tilde{N}(ds, dz) \\ &= \int_{(0,t]} \int_Z \gamma_\delta(u_{s-}^\epsilon(x) + K_s^\epsilon(z, x)) - \gamma_\delta(u_{s-}^\epsilon(x)) N(ds, dz) \\ &\quad + \int_0^t \int_Z \gamma_\delta(u_{s-}^\epsilon(x) + K_s^\epsilon(z, x)) - \gamma_\delta(u_{s-}^\epsilon(x)) \nu(dz) ds. \end{aligned}$$

Notice that if we integrate the above over  $x$ , we can interchange the integrals, since the first term at the right hand side is a finite sum, and for the second term it is obvious by the integrability conditions that we have by assumption. Therefore, integrating (2.5) over  $x$  we get

$$\int_{\mathbb{R}^d} \gamma_\delta(u_t^\epsilon(x)) dx = \int_{\mathbb{R}^d} \gamma_\delta(u_0^\epsilon(x)) dx + \int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x) dx ds$$



$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x) dx dw_s^k + \frac{1}{2} \sum_k \int_0^t \int_{\mathbb{R}^d} \alpha_\delta(u_s^\epsilon(x)) |h_t^{k\epsilon}(x)|^2 dx ds \\
& + \int_{(0,t]} \int_Z \int_{\mathbb{R}^d} \gamma_\delta(u_{s-}^\epsilon(x) + K_s^\epsilon(z, x)) - \gamma_\delta(u_{s-}^\epsilon(x)) dx \tilde{N}(ds, dz) \\
& + \int_0^t \int_Z \int_{\mathbb{R}^d} \gamma_\delta(u_s^\epsilon(x) + K_s^\epsilon(z, x)) \\
& - \gamma_\delta(u_s^\epsilon(x)) - \beta_\delta(u_s^\epsilon(x)) K_s^\epsilon(z, x) dx \nu(dz) ds. \tag{2.6}
\end{aligned}$$

We claim that letting a sequence  $\epsilon_l \rightarrow 0$  we get

$$\begin{aligned}
& \int_{\mathbb{R}^d} \gamma_\delta(u_t(x)) dx = \int_{\mathbb{R}^d} \gamma_\delta(u_0(x)) dx + \int_0^t \langle \beta_\delta(u_s(x)), v_s^* \rangle ds \\
& \int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x) dx dw_s^k + \frac{1}{2} \sum_k \int_0^t \int_{\mathbb{R}^d} \alpha_\delta(u_s(x)) |h_s^k(x)|^2 dx ds \\
& + \int_{(0,t]} \int_Z \int_{\mathbb{R}^d} \gamma_\delta(u_{s-}(x) + K_s(z, x)) - \gamma_\delta(u_{s-}(x)) dx \tilde{N}(ds, dz) \\
& + \int_0^t \int_Z \int_{\mathbb{R}^d} \gamma_\delta(u_s(x) + K_s(z, x)) \\
& - \gamma_\delta(u_s(x)) - \beta_\delta(u_s(x)) K_s(z, x) dx \nu(dz) ds. \tag{2.7}
\end{aligned}$$

We show that each of the terms in (2.6) converges to the corresponding ones in (2.7). For the left hand side, it suffices to show that for any sequence  $\epsilon_k \rightarrow 0$ , and for any subsequence  $\epsilon_{k_n}$  of it, there exists a subsequence of  $\epsilon_{k_n}$ , such that the convergence takes place. To this end, let  $\epsilon_k \rightarrow 0$ , and take a subsequence, denoted also by  $\epsilon_k$ . We fix  $(\omega, t)$ . We have that  $u_t^{\epsilon_k} \rightarrow u_t$  in  $L^2$ . By the equality  $(a^2 - b^2) = (a - b)(a + b)$  we see that  $(u_t^{\epsilon_k})^2 \rightarrow u_t^2$  in  $L^1$ . By the previous lemma, there exist  $g \in L^1$  and a further subsequence, denoted again  $\epsilon_k$ , such that

$$|u_t^{\epsilon_k}(x)|^2 \leq g(x) \quad \text{for all } x.$$

We have  $\gamma_\delta(u_t^{\epsilon_k}(x)) \rightarrow \gamma_\delta(u_t(x))$  for almost every  $x$ , and since

$$|\gamma_\delta(u_t^{\epsilon_k}(x))| \leq \frac{(u_t^{\epsilon_k}(x))^2}{2} \leq \frac{g(x)}{2},$$

we obtain

$$\int_{\mathbb{R}^d} \gamma_\delta(u_t^{\epsilon_k}(x)) dx \rightarrow \int_{\mathbb{R}^d} \gamma_\delta(u_t(x)) dx.$$

This implies that for the fixed  $(\omega, t) \in \Omega \times [0, T]$

$$\int_{\mathbb{R}^d} \gamma_\delta(u_t^\epsilon(x)) dx \rightarrow \int_{\mathbb{R}^d} \gamma_\delta(u_t(x)) dx.$$

To see the convergence of the second term in the right-hand side of (2.6) we fix  $(s, \omega)$  such that  $u_s \in V$ . Taking into account the well-known fact that there exist  $f_s^i \in L^2(\mathbb{R}^d)$  for  $i = 0, 1, \dots, d$  such that

$$v_s^* = f_s^0 + D_i f_s^i,$$

it is not difficult to see that

$$v_s^{*\epsilon} = f_s^{0\epsilon} + D_i f_s^{i\epsilon}.$$

Therefore

$$|v_s^* - v_s^{*\epsilon}|_{V^*} \leq \left( \sum_{i=0}^n |f_s^i - f_s^{i\epsilon}|_H^2 \right)^{1/2} \rightarrow 0.$$

It is also straightforward to check that

$$|\beta_\delta(u_s^\epsilon) - \beta_\delta(u_s)|_V \rightarrow 0.$$

Hence we conclude

$$\int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x) dx = \langle v_s^{*\epsilon}, \beta_\delta(u_s^\epsilon) \rangle \rightarrow \langle v_s^*, \beta_\delta(u_s) \rangle.$$

Notice now that for each  $\epsilon$  for almost every  $(\omega, s)$  we have

$$\left| \int_{\mathbb{R}^n} \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x) dx \right| \leq C(|u_s|_V^2 + |v_s^*|_{V^*}^2),$$

which is almost surely integrable on  $[0, T]$ . Therefore almost surely

$$\int_0^t \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) v_s^{*\epsilon}(x) dx ds \rightarrow \int_0^t \langle v_s^*, \beta_\delta(u_s) \rangle ds \quad \text{for all } t.$$

For the sum of the stochastic integrals against the Wiener processes we just note that for almost all  $(\omega, s)$

$$\sum_k \left| \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x) dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x) dx \right|^2 \rightarrow 0$$

and

$$\begin{aligned} \sum_k \left| \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x) dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x) dx \right|^2 \\ \leq 4 \sup_{t \leq T} |u_t|_{L^2}^2 \sum_k |h_s^k|_{L^2}^2 \end{aligned}$$

which is integrable on  $[0, T]$ . Therefore

$$\int_0^T \sum_k \left| \int_{\mathbb{R}^d} \beta_\delta(u_s^\epsilon(x)) h_s^{k\epsilon}(x) dx - \int_{\mathbb{R}^d} \beta_\delta(u_s(x)) h_s^k(x) dx \right|^2 ds \rightarrow 0$$

almost surely, which implies that the sum of the stochastic integrals converges in probability, uniformly in  $t$ , and there exists a sequence  $\epsilon_l$

such that the convergence happens almost surely. Note that for each  $k$  we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \alpha_\delta(u_s^\epsilon(x)) |h_s^{k\epsilon}(x)|^2 - \alpha_\delta(u_s(x)) |h_s^k(x)|^2 dx \right| \\ & \leq \int_{\mathbb{R}^d} |(h_s^{k\epsilon}(x))^2 - (h_s^k(x))^2| dx \\ & + \int_{\mathbb{R}^d} |h_s^k(x)|^2 |\alpha_\delta(u_s^\epsilon(x)) - \alpha_\delta(u_s(x))| dx \rightarrow 0. \end{aligned}$$

Moreover,

$$\left| \int_{\mathbb{R}^d} \alpha_\delta(u_s^\epsilon(x)) |h_s^{k\epsilon}(x)|^2 dx \right| \leq |h_s^k|_H^2,$$

where the right hand side is almost surely integrable on  $[0, T]$ . Hence the convergence of the fourth term in the right-hand side of (2.6) follows by Lebesgue's theorem on dominated convergence. Similar arguments show the convergence of the last two terms. We conclude that almost surely (2.7) holds for all  $t \in [0, T]$ .

Now by letting  $\delta \rightarrow 0$ , using arguments similar to the previous ones, and keeping in mind that

$$|\gamma_\delta(r)| \leq r^2/2, \quad |\beta_\delta(r)| \leq |r|, \quad |\alpha_\delta(r)| \leq 1$$

and that for all  $v \in V$

$$|\beta_\delta(v) - v|_V \rightarrow 0, \quad |\beta_\delta(v)|_V \leq |v|_V,$$

we can finish the proof of the theorem for  $Q = \mathbb{R}^d$ .

We reduce the case of a bounded Lipschitz domain  $Q$  to that of the whole space by using the sequence  $\phi_n$  from Lemma 2.3. Remember that  $\phi_n$  has compact support in  $Q$ . Thus for a function  $\eta$  on  $Q$  we denote also by  $\phi_n \eta$  the function on the whole  $\mathbb{R}^d$  which is zero outside of  $Q$  and agrees with  $\phi_n \eta$  on  $Q$ . Notice that when  $u$  satisfies (2.1) on  $Q$ , then  $\phi_n u$  satisfies

$$\begin{aligned} \phi_n u_t &= \phi_n u_0 + \int_0^t \phi_n v_s^* ds + \int_0^t \phi_n h_s^k dw_s^k \\ &+ \int_0^t \int_Y \phi_n K_s(z) \tilde{N}(ds, dz) \end{aligned}$$

on the whole  $\mathbb{R}^d$ , where the functional  $\phi_n v^*$  is defined by

$$\langle \phi_n v_s^*, g \rangle := \langle v_s^*, \phi_n g \rangle_Q$$

for  $g \in H^1(\mathbb{R}^d)$ . The notation  $\langle \cdot, \cdot \rangle_Q$  means the duality product between  $H_0^1(Q)$  and  $H^{-1}(Q)$ . Notice that  $\langle v_s^*, \phi_n g \rangle_Q$  is well defined, since the

restriction of  $\phi_n g$  to  $Q$  belongs to  $H_0^1(Q)$ . Then by the result in the case of the whole space we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \phi_n^2 |u_t^+|^2 dx = \int_{\mathbb{R}^d} |\phi_n u_0^+|^2 dx + 2 \int_0^t \langle v_s^*, \phi_n^2 u_s^+ \rangle ds \\
& + 2 \int_0^t \int_{\mathbb{R}^d} \phi_n^2 h_s^k u_s^+ dx dw_s^k + \int_0^t \sum_k \int_{\mathbb{R}^d} |\mathbb{I}_{\{\phi_n u_s > 0\}} \phi_n h_s^k|^2 dx ds \\
& + \int_{(0,t]} \int_Z \int_{\mathbb{R}^d} |\phi_n(u_{s-} + K_s(z))^+|^2 - |\phi_n u_{s-}^+|^2 dx \tilde{N}(ds, dz) \\
& + \int_0^t \int_Z \int_{\mathbb{R}^d} |\phi_n(u_s + K_s(z))^+|^2 - |\phi_n u_s^+|^2 - 2K_s(z) \phi_n^2 u_s^+ dx \nu(dz) ds,
\end{aligned}$$

which means

$$\begin{aligned}
& \int_Q \phi_n^2 |u_t^+|^2 dx = \int_Q |\phi_n u_0^+|^2 dx + 2 \int_0^t \langle v_s^*, \phi_n^2 u_s^+ \rangle ds \\
& + 2 \int_0^t \int_Q \phi_n^2 h_s^k u_s^+ dx dw_s^k + \int_0^t \int_Q \sum_k |\mathbb{I}_{\{\phi_n u_s > 0\}} \phi_n h_s^k|^2 dx ds \\
& + \int_{(0,t]} \int_Z \int_Q |\phi_n(u_{s-} + K_s(z))^+|^2 - |\phi_n u_{s-}^+|^2 dx \tilde{N}(ds, dz) \\
& + \int_0^t \int_Z \int_Q |\phi_n(u_s + K_s(z))^+|^2 - |\phi_n u_s^+|^2 - 2K_s(z) \phi_n^2 u_s^+ dx \nu(dz) ds.
\end{aligned}$$

It is now easy to take  $n \rightarrow \infty$  here to finish the proof of the theorem. We only note that for the second term on the right-hand side we have by Lemma 2.3 and Remark 2.1

$$\langle v_s^*, \phi_n^2 u_s^+ \rangle \rightarrow \langle v_s^*, u_s^+ \rangle \quad \text{for all } \omega, s,$$

and for a constant  $C$ ,

$$\langle v_s^*, \phi_n^2 u_s^+ \rangle \leq C |v_s^*|_{V^*} |u_s|_V \quad \text{for all } n.$$

□

### 3. THE COMPARISON PRINCIPLE

Now we present our comparison principle for equation (1.1). We make the following assumptions. Let  $K$  denote a constant.

**Assumption 3.1.**

i) The coefficients  $a^{ij}$ ,  $b^i$  and  $c$  are real-valued  $\mathcal{P} \times \mathcal{B}(Q)$  measurable functions on  $\Omega \times [0, T] \times Q$  and are bounded by  $K$  for every  $i, j = 1, \dots, d$ . The coefficient  $\phi^i = (\phi^{ik})_{k=1}^\infty$  is an  $l_2$ -valued  $\mathcal{P} \times \mathcal{B}(Q)$ -measurable function such that

$$\sum_i \sum_k |\phi_t^{ik}(x)|^2 \leq K \quad \text{for all } \omega, t \text{ and } x,$$

ii)  $f$  and  $\sigma = (\sigma^k)_{k=1}^\infty$  are  $\mathcal{P} \times \mathcal{B}(Q) \times \mathcal{B}(\mathbb{R})$ -measurable functions on  $\Omega \times [0, T] \times Q \times \mathbb{R}$ , with values in  $\mathbb{R}$  and in  $l_2$ , respectively. The function  $g$  is defined on  $\Omega \times [0, T] \times Q \times Z \times \mathbb{R}$  with values in  $\mathbb{R}$  and it is  $\mathcal{P} \times \mathcal{B}(Q) \times \mathcal{Z} \times \mathcal{B}(\mathbb{R})$ -measurable. We assume that there exists  $h \in L^2(\Omega \times [0, T] \times Q)$ , predictable as a process with values in  $L^2$  such that for all  $\omega, t, x, z, r$ ,

$$|f_t(x, r)|^2 + \sum_k |\sigma_t^k(x, r)|^2 + \int_Z |g_t(x, z, r)|^2 \nu(dz) \leq K|r|^2 + |h_t(x)|^2,$$

iii)  $\psi$  is an  $\mathcal{F}_0$ -measurable random variable in  $L^2(Q)$  and we have  $E|\psi|_{L^2}^2 < \infty$ ,

iv) there exists a constant  $\theta > 0$  such that for all  $\omega, t, x$  and for all  $\zeta = (\zeta_1, \dots, \zeta_d) \in Q$  we have

$$a_t^{ij}(x)\zeta_i\zeta_j - \frac{1}{2}\phi_t^{ik}(x)\phi_t^{jk}(x)\zeta_i\zeta_j \geq \theta|\zeta|^2,$$

v) for all  $\omega, t, x, z, r_1, r_2$

$$\begin{aligned} & (r_1 - r_2)(f_t(x, r_1) - f_t(x, r_2)) + \sum_k |\sigma_t^k(x, r_1) - \sigma_t^k(x, r_2)|^2 \\ & + \int_Z |g_t(x, z, r_1) - g_t(x, z, r_2)|^2 \nu(dz) \leq K|r_1 - r_2|^2 \end{aligned}$$

vi)  $f$  is continuous in  $r$ .

**Assumption 3.2.** The function  $r + g_t(x, z, r)$  is non-decreasing in  $r$  for all  $\omega, t, x, z$ .

A solution of equation (1.1) is understood in the generalized sense, defined, e.g., in [6]. This means it is a strongly cadlag adapted process  $u$  with values in  $L^2(Q)$  such that

i)  $u_t \in H_0^1(Q)$  for  $dP \times dt$  almost every  $(\omega, t) \in \Omega \times [0, T]$

ii)  $E \int_0^T |u_t|_{H_0^1}^2 dt < \infty$

iii) for all  $v \in H_0^1(Q)$  we have almost surely, for all  $t \in [0, T]$

$$\begin{aligned}
(u_t, v) &= (\psi, v) + \int_0^t \{-(a_s^{ij} D_i u_s D_j v) + (b_s^i u_s, v) \\
&\quad + (c_s u_s, v) + (f_s(u_s), v)\} ds \\
&\quad + \int_0^t \{(\phi_s^{ik} D_i u_s, v) + (\sigma_s^k(u_s), v)\} dw_s^k \\
&\quad + \int_{(0,t]} \int_Z (g_s(z, u_{s-}), v) \tilde{N}(dz, ds),
\end{aligned}$$

where  $(\cdot, \cdot)$  is the inner product in  $L^2(Q)$ .

*Remark 3.1.* Under Assumption 3.1 there exist a unique solution of equation (1.1). This follows by the fact that under Assumption 3.1 the equation satisfies the assumptions in [6] (which will be mentioned later as Assumption 3.3) in the abstract setting.

We are now ready to state our main result.

**Theorem 3.1.** *Let  $u$  and  $v$  satisfy the equations*

$$\begin{aligned}
du_t(x) &= \{\mathcal{L}_t u_t(x) + f_t(x, u_t)\} dt + \{\phi_t^{ik}(x) D_i u_t(x) + \sigma_t^k(x, u_t)\} dw_t^k \\
&\quad + \int_Z g_t(x, z, u_{t-}) \tilde{N}(dt, dz), \\
u_0(x) &= \psi(x),
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
dv_t(x) &= \{\mathcal{L}_t v_t(x) + F_t(x, v_t)\} dt + \{\phi_t^{ik}(x) D_i v_t(x) + \sigma_t^k(x, v_t)\} dw_t^k \\
&\quad + \int_Z g_t(x, z, v_{t-}) \tilde{N}(dt, dz), \\
v_0(x) &= \Psi(x)
\end{aligned} \tag{3.2}$$

Suppose that Assumptions 3.1, 3.2 are satisfied for each of the equations. Let  $f \leq F$  and  $\psi \leq \Psi$ . Then for all  $t \in [0, T]$  we have

$$E|(u_t - v_t)^+|^2_{L^2(Q)} = 0.$$

**Corollary 3.2.** *If the assumptions of the theorem are satisfied, then almost surely for all  $t \in [0, T]$  we have that  $u_t(x) \leq v_t(x)$  for almost every  $x \in Q$ .*

We will first prove the following lemma.

**Lemma 3.3.** *Let  $Y \subset Z$ ,  $\nu(Y) < \infty$  and suppose that  $u$  and  $v$  satisfy the equations*

$$\begin{aligned} du_t(x) &= \{\mathcal{L}_t u + f_t(x, u_t)\}dt + \{\phi_t^{ik}(x)D_i u_t + \sigma_t^k(x, u_t)\}dw_t^k \\ &\quad + \int_Y g_t(x, z, u_{t-})\tilde{N}(dt, dz), \\ u_0 &= \psi, \end{aligned} \tag{3.3}$$

$$\begin{aligned} dv_t(x) &= \{\mathcal{L}_t v + F(t, x, v)\}dt + \{\phi_t^{ik}(x)D_i v_t(x) + \sigma_t^k(x, v_t)\}dw_t^k \\ &\quad + \int_Y g_t(x, z, v_{t-})\tilde{N}(dt, dz), \\ v_0 &= \Psi. \end{aligned} \tag{3.4}$$

Let *i*) through *iv*) from Assumption 3.1 hold for the two equations and assume that either  $f$  or  $F$  satisfies also *v*). Let Assumption 3.2 also hold and assume  $f \leq F$  and  $\psi \leq \Psi$ . Then for all  $t \in [0, T]$  we have

$$E|(u_t - v_t)^+|_{L^2(Q)}^2 = 0.$$

*Proof.* Without loss of generality we can assume that *v*) is satisfied by  $f$ . For the difference  $h = u - v$  we have

$$\begin{aligned} h_t &= h_0 + \int_0^t \mathcal{L}_s h_s + f_s(u_s) - F_s(v_s) ds \\ &\quad + \int_0^t \phi_s^{ki} D_i h_s + \sigma_s^k(u_s) - \sigma_s^k(v_s) dw_s^k \\ &\quad + \int_{(0,t]} \int_Y g(s, z, u_{s-}) - g(s, z, v_{s-}) \tilde{N}(ds, dz). \end{aligned}$$

By Theorem 2.1 we have

$$\begin{aligned} |h_t^+|_{L^2}^2 &= 2 \int_0^t \int_Q \left\{ -a_s^{ij}(x) D_i h_s^+(x) D_j h_s^+(x) \right. \\ &\quad \left. + b_s^i(x) D_i h_s^+(x) h_s^+(x) + c_s(x) |h_s^+(x)|^2 + (f_s(x, u_s) - F_s(x, v_s)) h_s^+(x) \right. \\ &\quad \left. + \sum_k \left| \mathbb{I}_{h_s \geq 0} \sum_i \phi_s^{ki}(x) D_i h_s(x) + \mathbb{I}_{h_s \geq 0} (\sigma_s^k(x, u_s(x)) - \sigma_s^k(x, v_s(x))) \right|^2 \right\} dx ds \\ &\quad + \int_0^t \int_Y \int_Q \{ [h_s(x) + g_s(x, z, u_{s-}(x)) - g_s(x, z, v_{s-}(x))]^+ \}^2 \\ &\quad - |h_s(x)^+|^2 - 2h_s^+(x) [g_s(x, z, u_s(x)) - g_s(x, z, v_s(x))] dx \nu(dz) ds + m_t \end{aligned}$$

for a martingale  $m$ . Notice that

$$\begin{aligned} &(f_s(x, u_s(x)) - F_s(x, v_s(x))) h_s^+(x) \\ &= (f_s(x, u_s(x)) - F_s(x, v_s(x)) + F_s(x, v_s(x)) - F_s(x, v_s(x))) h_s^+(x) \end{aligned}$$

$$\leq [f_s(x, u_s(x)) - F_s(x, v_s(x))]h_s^+(x).$$

Using this, the inequality  $|ab| \leq (1/c)a^2 + cb^2$ , iv) of Assumption 3.1 and the boundness of the coefficients, we have for some constant  $C$  depending on  $\theta$ ,  $K$  and  $d$ , that

$$\begin{aligned} |h_t^+|_{L^2(Q)}^2 &\leq K \int_0^t |h_s^+|_{L^2(Q)}^2 ds \\ &+ K \int_0^t \int_Q \{[f_s(x, u_s(x)) - F_s(x, v_s(x))]h_s^+(x) + \mathbb{I}_{h_s > 0} |\sigma_s^k(x, u_s(x)) - \sigma_s^k(x, v_s(x))|^2 ds \\ &\quad + \int_0^t \int_Y \int_Q \{[h_s(x) + g_s(x, z, u_{s-}(x)) - g_s(x, z, v_{s-}(x))]^+ \}^2 \\ &\quad - |h_s^+(x)|^2 - 2h_s^+(x)[g_s(x, z, u_s(x)) - g_s(x, z, v_s(x))] dx \nu(dz) ds + m_t. \end{aligned}$$

Taking expectation and using Assumption 3.2 we obtain

$$\begin{aligned} E|h_t^+|_{L^2(Q)}^2 &\leq C \int_0^t E|h_s^+|_{L^2(Q)}^2 ds \\ &+ CE \int_0^t \int_Q \{[f_s(x, u_s(x)) - f_s(x, v_s(x))]h_s^+(x) + \mathbb{I}_{h_s > 0} |\sigma_s^k(x, u_s(x)) - \sigma_s^k(x, v_s(x))|^2 \\ &\quad + \mathbb{I}_{h_s > 0} \int_Y |g_s(x, z, u_s(x)) - g_s(x, z, v_s(x))|^2 \nu(dz) \} dx ds. \end{aligned}$$

Hence due to v) of Assumption 3.1 we have a constant  $C$  such that

$$E|h_t^+|_{L^2(Q)}^2 \leq C \int_0^t E|h_s^+|_{L^2(Q)}^2 ds \quad \text{for all } t \in [0, T],$$

and the result follows by Gronwall's lemma.  $\square$

We will need to approximate the solutions of equations (3.1) and (3.2) by solutions of equations of the type (3.3) and (3.4), respectively. We formulate the corresponding statement, Lemma 3.4 in an abstract setting below. To this end we consider a separable reflexive Banach space  $V$ , embedded continuously and densely into a Hilbert space  $H$ , which is identified with its dual  $H^*$  by the help of the inner product in  $H$ . Thus we have

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

where  $H^* \hookrightarrow V^*$ , the adjoint of  $V \hookrightarrow H$ , is also a bounded and dense embedding. We consider the operators

$$\mathcal{A} : \Omega \times [0, T] \times V \rightarrow V^*, \quad \mathcal{B}^k : \Omega \times [0, T] \times V \rightarrow H,$$

$$\mathcal{C} : \Omega \times [0, T] \times Z \times V \rightarrow H$$



for integers  $k \geq 1$ , where  $\mathcal{A}$  and  $\mathcal{B}^k$  are  $\mathcal{P} \times \mathcal{B}(V)$  measurable and  $\mathcal{C}$  is  $\mathcal{P} \times \mathcal{Z} \times \mathcal{B}(V)$  measurable. We state now the conditions under which there exists a unique solution to the equation (see [6], [7]),

$$\begin{aligned} du_t &= \mathcal{A}(t, u_t)dt + \mathcal{B}^k(t, u_t)dw_t^k + \int_Z \mathcal{C}(t, z, u_{t-})\tilde{N}(dt, dz), \\ u_0 &= \psi. \end{aligned}$$

**Assumption 3.3.** There exist constants  $L \geq 0$ ,  $\epsilon > 0$ , a nonnegative predictable process  $g$  such that for all  $(\omega, t)$  and all  $v, v_1, v_2 \in V$

- i)  $\langle v, \mathcal{A}(t, v_1 + \lambda v_2) \rangle$  is continuous in  $\lambda$  on  $\mathbb{R}$ ,
- ii)  $2\langle v_1 - v_2, \mathcal{A}(t, v_1) - \mathcal{A}(t, v_2) \rangle + \sum_k |\mathcal{B}^k(t, v_1) - \mathcal{B}^k(t, v_2)|_H^2 + \int_Z |\mathcal{C}(t, z, v_1) - \mathcal{C}(t, z, v_2)|_H^2 \nu(dz) \leq L|v_1 - v_2|_H^2$ ,
- iii)  $2\langle v, \mathcal{A}(t, v) \rangle + \sum_k |\mathcal{B}^k(t, v)|_H^2 + \int_Z |\mathcal{C}(t, v, z)|_H^2 \nu(dz) + \epsilon|v|_V^2 \leq g(t) + L|v|_H^2$ ,
- iv)  $|\mathcal{A}(t, v)|_{V^*}^2 \leq g(t) + L|v|_V^2$
- v)  $E|\psi|^2 < \infty$ ,  $E \int_0^T g(t)dt < \infty$ .

We have the following lemma.

**Lemma 3.4.** Let  $u, u^n$ , for integers  $n \geq 1$ , be the solutions of the equations

$$\begin{aligned} du_t &= \mathcal{A}(t, u_t)dt + \mathcal{B}^k(t, u_t)dw_t^k + \int_Z \mathcal{C}(t, z, u_{t-})\tilde{N}(dt, dz), \\ du_t^n &= \mathcal{A}(t, u_t^n)dt + \mathcal{B}^k(t, u_t^n)dw_t^k + \int_{Z^n} \mathcal{C}(t, z, u_{t-}^n)\tilde{N}(dt, dz) \end{aligned}$$

respectively, with common initial condition  $\psi$ , where  $Z_n \in \mathcal{Z}$ . Let Assumption 3.3 hold, and assume that  $\mathbb{I}_{Z_n} \rightarrow \mathbb{I}_Z$  for  $n \rightarrow \infty$ . Then  $E|u_t^n - u_t|^2 \rightarrow 0$  for each  $t \in [0, T]$ .

*Proof.* For the difference we have

$$\begin{aligned} u_t - u_t^n &= \int_0^t \mathcal{A}(s, u_s) - \mathcal{A}(s, u_s^n)ds \\ &\quad + \int_0^t \mathcal{B}^k(s, u_s) - \mathcal{B}^k(s, u_s^n)dw_s^k + \\ &\quad \int_{(0,t]} \int_Z \{\mathcal{C}(s, u_{s-}, z) - \mathcal{C}(s, u_{s-}^n, z)\} \mathbb{I}_{Z_n} + \mathbb{I}_{Z-Z_n} \mathcal{C}(s, u_{s-}, z) \tilde{N}(ds, dz) \end{aligned}$$

By Itô's formula (see [5]) we get

$$|u_t - u_t^n|_H^2 = 2 \int_0^t \langle u_s - u_s^n, \mathcal{A}(s, u_s) - \mathcal{A}(s, u_s^n) \rangle ds$$

$$\begin{aligned}
& + \int_0^t \sum_k |\mathcal{B}^k(s, u_s) - \mathcal{B}^k(s, u_s^n)|_H^2 ds + m_t \\
& + \int_0^t \int_Z |\{\mathcal{C}(s, u_{s-}, z) - \mathcal{C}(s, u_{s-}^n, z)\} \mathbb{I}_{Z_n} \\
& \quad + \mathbb{I}_{Z-Z_n} \mathcal{C}(s, u_{s-}, z)|_H^2 \nu(dz) ds \\
& = \int_0^t \langle u_s - u_s^n, \mathcal{A}(s, u_s) - \mathcal{A}(s, u_s^n) \rangle ds \\
& + m_t + \int_0^t \sum_k |\mathcal{B}^k(s, u_s) - \mathcal{B}^k(s, u_s^n)|_H^2 ds \\
& + \int_0^t \int_Z |\{\mathcal{C}(s, u_s, z) - \mathcal{C}(s, u_s^n, z)\} \mathbb{I}_{Z_n}|_H^2 \\
& \quad + |\mathbb{I}_{Z-Z_n} \mathcal{C}(s, u_s, z)|_H^2 \nu(dz) ds
\end{aligned}$$

with a martingale  $m$ . Now taking expectations and using ii) from Assumption 3.3 we have

$$E|u_t - u_t^n|_H^2 \leq L \int_0^t E|u_s - u_s^n|_H^2 ds + A_n,$$

where

$$A_n = \int_0^T \int_Z \mathbb{I}_{Z-Z_n} E|\mathcal{C}(s, u_s, z)|_H^2 \nu(dz) ds.$$

Then by Gronwall's lemma,

$$E|u_t - u_t^n|_H^2 \leq A_n e^{LT}.$$

It is clear that  $A_n \rightarrow 0$ , and the result follows.  $\square$

*Proof of Theorem 3.1.* Let  $Z_n \in \mathcal{Z}$ , such that  $\nu(Z_n) < \infty$  and  $\cup_n Z_n = Z$ . Since Assumptions 3.1 and 3.2 are satisfied, the equations

$$\begin{aligned}
du_t^n(x) &= \{\mathcal{L}_t u_t^n(x) + f_t(x, u_t^n(x))\} dt \\
& + \{\phi_t^{ik}(x) D_i u_t^n(x) + \sigma_t^k(x, u_t^n(x))\} dw_t^k \\
& + \int_{Z_n} g_t(x, z, u_{t-}^n(x)) \tilde{N}(dt, dz)
\end{aligned}$$

and

$$\begin{aligned}
dv_t^n(x) &= \{\mathcal{L}_t v_t^n(x) + F_t(x, v_t^n(x))\} dt \\
& + \phi_t^{ik}(x) D_i v_t^n(x) + \sigma_t^k(x, v_t^n(x))\} dw_t^k \\
& + \int_{Z_n} g_t(x, z, v_{t-}^n(x)) \tilde{N}(dt, dz),
\end{aligned}$$

with initial condition  $u_0^n = \phi$  and  $v_0^n = \Psi$ , respectively, admit a unique solution  $u^n$  and  $v^n$  for each  $n$ . Then by Lemmas 3.3 and 3.4 we have

$$E|(u_t - v_t)^+|_{L^2(Q)}^2 = \lim_{n \rightarrow \infty} E|(u_t^n - v_t^n)^+|_{L^2(Q)}^2 = 0.$$

□

*Remark 3.2.* Assumption 3.2 cannot be omitted in Theorem 3.1. Consider for example the SDE

$$u_t = 1 - \int_{(0,t]} 2u_{s-} d\tilde{N}_s,$$

where  $N_t$  is a Poisson process with intensity one. Let  $\tau$  be the time that the first jump of  $N$  occurs. Then  $P(\tau \leq T) > 0$ . Since  $u_t = e^{-2t}$  on  $[0, \tau)$ , one can see that on the set  $\{\tau \leq T\}$  we have  $u(\tau) = -e^{-2\tau} < 0$ .

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SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, KING'S BUILDINGS,  
EDINBURGH, EH9 3JZ, UK